

UR-1259
ER-40685-711
CTP TAMU-31/92
June 1992

Self-Duality in 3+3 Dimensions and the KP Equation

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ABSTRACT

We consider two types of generalized self-duality conditions for Yang-Mills fields on paracomplex manifolds of arbitrary dimension. We then specialize to $3 + 3$ dimensions and show how one can obtain the KP equation from these self-duality conditions on $SL(2, R)$ valued gauge fields.

* Supported in part by the U.S. Department of Energy, under grant DE-AC02-76ER-13065.

† Supported in part by the U.S. National Science Foundation, under grant PHY-9106593.

1. Introduction

There are many physical systems that are described by nonlinear, Hamiltonian equations which are integrable, i.e., they can be solved exactly [1,2]. These systems possess many interesting properties which make them quite attractive. In recent years, a growing number of connections have been found between such systems and some of the models of theoretical high- energy physics. For example, it is known that the various conformal algebras [3,4] (W_N algebras etc.) arise as the Hamiltonian structures (Poisson brackets) of the integrable systems. In fact, the first Hamiltonian structure [5] of the 2+1 dimensional integrable system, the KP hierarchy [6], can be identified with the W_∞ algebra [7] which has been a subject of much recent work in diverse areas, e.g. in 2D gravity [8].

In a separate development, it was found that the 1+1 dimensional integrable systems can be obtained from the self-duality conditions imposed on the Yang-Mills potentials in four and higher dimensions [9]. Thus, for example, it is known that the KdV equation, the nonlinear Schrödinger equation, etc., can be obtained from the self-duality condition on the $SL(2,R)$ valued Yang-Mills potentials in four dimensions with (2,2) signature upon appropriate reduction [10]. Similarly, the generalized KdV equations (namely, Boussinesq equation etc.) can also be obtained from the self-duality condition for higher $SL(N,R)$ groups [11]. It is also known that the KP hierarchy leads to most of the 1+1 dimensional integrable models upon appropriate reduction. It is, therefore, quite natural to expect that the KP equation can also be obtained from a self-duality condition of the Yang-Mills potential in higher dimensions. Various attempts in this direction, however, have been unsuccessful and the reason for the failure is normally attributed to the nonlocal nature of the scalar Lax operator for the KP equation.

The derivation of various 1+1 or 2+1 dimensional integrable systems from self-duality conditions in higher dimensions provides a unified picture for seemingly unrelated vast class of integrable systems. As a byproduct, it may in principle give rise to new integrable systems. Moreover, the symmetries of all these integrable systems can now be understood in terms of gauge transformations in higher dimensions. This not only provides a way to understand these symmetries, but also provides a way to search for new hidden symmetries, if any.

In this letter, inspired by the successful embedding of the KdV equation in the self-duality condition on the Yang-Mills potential in 2+2 dimensions, which we shall outline in Section 2, we analyze the self-duality conditions for Yang-Mills potentials in 3+3 dimensions and show how the KP equation follows from such a condition on $SL(2,R)$ valued Yang-Mills potentials. In fact, in Section 3, we shall describe two types of generalized self-duality conditions on paracomplex manifolds [12] of arbitrary dimension, and then specialize to 3 + 3 dimensions. In Section 4, we shall derive the KP equation from these self-duality equations. In the last section, we shall comment on open problems.

2. Self-Duality in 2+2 Dimensions and the KdV Equation

In this section we shall outline the embedding of KdV equation in self-duality condition in 2+2 dimensions [10,11]. Consider the SO(2,2) invariant line element

$$ds^2 = 2dxdy + 2dzdt. \quad (1)$$

The metric $\eta_{\mu\nu}$ is defined by $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$, where $x^\mu = (x, y, z, t)$, and evidently has the signature $(+, -, +, -)$. The self-duality condition is

$$F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \quad (2)$$

where the indices of the field strength $F_{\mu\nu}$ for the Yang-Mills potential A_μ are raised and lowered by the metric $\eta_{\mu\nu}$. Eq. (2) is equivalent to the following three equations (with $\epsilon_{xyzt} = -1$)

$$F_{tx} = 0, \quad (3)$$

$$F_{yz} = 0, \quad (4)$$

$$F_{tz} + F_{xy} = 0. \quad (5)$$

Let us identify t, x as the coordinates of the 1+1 dimensional spacetime, and use the notation $A_t = H$, $A_x = Q$, $A_y = P$, $A_z = -B$. Imposing the conditions

$$\partial_z = 0, \quad \partial_y - \partial_x = 0, \quad (6)$$

and taking B to be constant, the self-duality equations (3-5) reduce to

$$[\partial_t - H, \partial_x - Q] = 0, \quad (7)$$

$$[P, B] = 0, \quad (8)$$

$$[H, B] = [\partial_x - Q, \partial_x - P]. \quad (9)$$

We need to make further ansatz for the gauge potentials in order to derive particular integrable systems from (7-9). Let the gauge potentials take their value in SL(2,R), and consider the following ansatz

$$B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} \lambda & 1 \\ -u & -\lambda \end{pmatrix}, \quad (10)$$

where u is an arbitrary function of $t, x + y$ and λ is an arbitrary constant. In terms of the matrices

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (11)$$

which obey the SL(2,R) algebra

$$[\sigma_+, \sigma_-] = \sigma_3, \quad [\sigma_3, \sigma_\pm] = \pm 2\sigma_\pm, \quad (12)$$

expanding the functions $H = H_3\sigma_3 + H_-\sigma_+ + H_+\sigma_-$ and $P = P_3\sigma_3 + P_-\sigma_+ + P_+\sigma_-$, first, from (8) one determines that

$$P_- = 0, \quad P_3 = 0. \quad (13)$$

Next, from (9) one learns that

$$H_- = -P_+, \quad H_3 = -\frac{1}{2}(u + P_+)' - \lambda P_+. \quad (14)$$

The functions H_+ and P_+ are still undetermined. Using (10) and (14) in the flatness condition (7), on the other hand, we obtain the equations

$$(u + 2P_+)' = 0, \quad (15)$$

$$H_+ = uP_+ - \lambda P_+' - \frac{1}{2}(u + P_+)'', \quad (16)$$

$$\dot{u} = \frac{1}{2}(u + P_+)''' + (u - P_+)u' + 2\lambda^2 P_+'. \quad (17)$$

From (15), setting the integration constant equal to zero we obtain $P_+ = -\frac{1}{2}u$, and consequently H and P are now completely determined in terms of a single function $u(t, x + y)$ and (17) now reads

$$\dot{u} = \frac{1}{4}u''' + \frac{3}{2}uu' - \lambda^2 u'. \quad (17a)$$

Shifting u by a constant as $u \rightarrow u + \frac{2}{3}\lambda^2$, this equation becomes identical to the usual KdV equation:

$$\dot{u} = \frac{1}{4}u''' + \frac{3}{2}uu', \quad (17b)$$

where t and $(x + y) \equiv x_+$, can be viewed as the time and space coordinates of a 1+1 dimensional spacetime and $'$ now denotes $\partial/\partial x_+$.

In summary, with the reduction conditions (6) and choices (10) for the $SL(2, \mathbb{R})$ potentials, the self-duality condition (2) leads to the KdV equation. It turns out that different choices for the connection Q lead to different integrable systems in 1+1 dimensions. However, a hierarchy of integrable systems such as the KdV hierarchy does not seem to follow from the self-duality equations. In the case of KdV hierarchy, the flatness condition (7) alone can accommodate it by suitable choices of H and Q , but those choices would be incompatible with the remaining equations (8) and (9) except for the case of KdV equation.

As mentioned earlier, it is desirable to obtain also the KP equation from a self-duality condition in a higher dimension. Interestingly enough, if we don't shift away the λ term in (17a), but instead take the ∂_x derivative of both sides and use the constraint $\partial_x u = \partial_y u$ in the λ dependent term, we obtain the KP equation described in Sec. 4 (after rescaling $u \rightarrow 2u$ and setting $\lambda = \sqrt{3}/2^\dagger$). Note, however, that the constraint $\partial_x u = \partial_y u$ is imposed by hand. Furthermore, it implies a 1+1 dimensional structure for the KP equation. In Sec.

[†] This procedure yields the KP equation (37). By choosing the integration constant arising from (15) appropriately, we can obtain also the KP equation (36).

4, we shall show how such a constraint as well as the KP equation arises from certain types of self-duality conditions in 3+3 dimensions. We shall then consider an alternative choice of gauge potentials which will lead to the KP equation, without the constraint $\partial_x u = \partial_y u$. We now turn to the description of these self-duality conditions.

3. Self-Duality Conditions in 3+3 Dimensions

Linear relationships among the components of the field strength $F_{\mu\nu}$ in d -dimensional Euclidean space of the following type

$$\lambda F_{\mu\nu} = \frac{1}{2} T_{\mu\nu\rho\sigma} F^{\rho\sigma}, \quad (19)$$

where the tensor $T_{\mu\nu\rho\sigma}$ is totally antisymmetric, and λ is a constant, have been considered in [13]. In four dimensions one can choose the T-tensor to be the Levi-Civita symbol which is $SO(4)$ invariant. However, in higher than four dimensions an $SO(d)$ invariant T-tensor is not available. Nonetheless, in some cases one can find a T-tensor which is invariant under a subgroup of $SO(d)$. Apart from this, the iteration of (19), of course, puts a severe restriction on the T-tensor. A number of examples in $d \leq 8$ have been provided in [13]. Further examples are given and the issue of which self-duality conditions can be deduced from the integrability condition of a linear equation was studied in [14].

Here we shall consider self-duality condition of the form (19) in a six dimensional space of signature (3,3). This is the simplest case beyond four dimensions in which we may seek the analogs of the (2,2) signature self-duality equations of the previous section. We shall also consider another type of self-duality equation in which the T-tensor is not required to be totally antisymmetric.

Both of these self-duality conditions will be formulated on paracomplex manifolds [12]. A paracomplex manifold M is a manifold which admits a (1,1) type tensor J_μ^ν which satisfies the condition

$$J_\mu^\nu J_\nu^\rho = \delta_\mu^\rho. \quad (20)$$

If M admits a metric $g_{\mu\nu}$ which satisfies the condition

$$J_\mu^\rho g_{\rho\nu} = -J_\nu^\rho g_{\rho\mu}, \quad (21)$$

then g will be called parahermitian metric, and M a parahermitian manifold.

Let M be a parahermitian manifold of dimension d . Our first self-duality condition on the field strength $F_{\mu\nu}$ is

$$\text{Case 1 :} \quad F_{\mu\nu} = \frac{3}{2} J_{[\mu\nu} J_{\rho\sigma]} F^{\rho\sigma}, \quad (22)$$

where $J_{\mu\nu}$ is the antisymmetric tensor defined by $J_{\mu\nu} = J_\mu^\rho g_{\rho\nu}$ and the antisymmetrizations are with unit strength, e.g. $J_{[\mu\nu} J_{\rho\sigma]} = \frac{1}{3}(J_{\mu\nu} J_{\rho\sigma} - J_{\mu\rho} J_{\nu\sigma} + J_{\mu\sigma} J_{\nu\rho})$. Iterating (22) one

finds that it is consistent by itself only in $d = 4$, while in dimensions other than four it is equivalent to the following equations

$$F_{\mu\nu} = -J_\mu{}^\rho J_\nu{}^\sigma F_{\rho\sigma} \quad \text{and} \quad J^{\mu\nu} F_{\mu\nu} = 0. \quad (23)$$

Let us now specialize to a flat six dimensional space of signature (3,3). We can choose the metric to be

$$g_{\mu\nu} = \mathbb{I}_3 \otimes \sigma_1. \quad (24)$$

In this coordinate system we can put the paracomplex structure in the following form [†]

$$J_\mu{}^\nu = \mathbb{I}_3 \otimes \sigma_3. \quad (25)$$

With these choices, the equations (22) and (23) read

$$F_{13} = 0, \quad F_{16} = 0, \quad F_{36} = 0, \quad (26)$$

$$F_{24} = 0, \quad F_{25} = 0, \quad F_{45} = 0, \quad (27)$$

$$F_{12} + F_{34} + F_{65} = 0. \quad (28)$$

These equations are rather similar to a set given in [13] in the case of Euclidean signature. In that case the invariance group of the self duality equations is $SU(3) \times U(1)$, while in our case the invariance group is $SL(3, R) \times SO(1, 1)$. In both cases, it is not known if these equations follow from the integrability conditions of a set of linear equations.

We now turn to the description of another type of self-duality condition which we can interpret as the integrability condition for a linear equation in a paracomplex manifold, M , in general. We shall subsequently specialize to the flat 3+3 dimensional space. The self-duality condition amounts to stating that $F_{\mu\nu}$ is pure, in the sense that given the projectors $P_{\pm\mu}{}^\nu = \frac{1}{2}(\delta_\mu{}^\nu \pm J_\mu{}^\nu)$, it satisfies, for example, $P_{+\mu}{}^\rho P_{+\mu}{}^\sigma F_{\rho\sigma} = 0$. This can be written as

$$\text{Case 2 :} \quad F_{\mu\nu} = -(J_\mu{}^\rho J_\nu{}^\sigma + 2J_\mu{}^\rho \delta_\nu^\sigma) F_{\rho\sigma}. \quad (29)$$

The iteration of this condition gives a consistent result, and no new conditions arise. If we view this condition in the form (19), the T-tensor here is not totally antisymmetric, and hence this falls outside the class of self-duality equations considered in [13]. Note furthermore, that the metric does not occur in (29), and therefore the symmetry group of this equation is huge, namely, all the paraholomorphic transformations of M , i.e. those coordinate transformations which preserve the form of the paracomplex structure J .

The self-duality condition (29) can be easily seen to follow as the integrability condition of the following linear equation

$$(\delta_\mu{}^\nu + J_\mu{}^\nu) D_\nu \psi = 0, \quad (30)$$

[†] We could have chosen the metric to take the form (25) in which case the paracomplex structure must take the form (24).

where ψ is an arbitrary function in some representation of the Yang-Mills gauge group, and the gauge covariant derivative is $D_\mu = \partial_\mu - A_\mu$. If we specialize to a space of signature (3,3), using the fact that it is possible to have a Majorana-Weyl spinor in this space, and some γ matrix identities special to this case, we can show that the linear equation (30) is equivalent to the following

$$\gamma^\mu \eta D_\mu \psi = 0, \quad (31)$$

where η is a commuting Majorana-Weyl spinor, and γ^μ are the Dirac γ -matrices in 3+3 dimensions obeying the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. To show this, consider another commuting Majorana-Weyl spinor of opposite chirality χ which is normalized such that $\bar{\chi}\eta = 1$. Multiplying (31) from the left with $\bar{\chi}\gamma_\nu$ we obtain

$$\bar{\chi}\gamma_\nu \gamma^\mu \eta D_\mu \psi = 0 \Rightarrow (\delta_\nu^\mu + \bar{\chi}\gamma_\nu^\mu \eta) D_\mu \psi = 0. \quad (32)$$

It remains to show that

$$\bar{\chi}\gamma_\nu^\mu \eta = J_\nu^\mu. \quad (33)$$

Indeed, by Fierz rearrangement we find that

$$\begin{aligned} J_\nu^\mu J_\mu^\rho &= \bar{\chi}\gamma_\nu^\mu \eta \bar{\chi}\gamma_\mu^\rho \eta \\ &= \frac{5}{4}\delta_\nu^\rho - \frac{1}{8}J^2 - J_\nu^\mu J_\mu^\rho, \end{aligned} \quad (34)$$

where $J^2 = J_\nu^\mu J_\mu^\nu$. Taking the trace of this equation one finds $J^2 = 6$, and substituting this back to (34) one finds that indeed $J_\nu^\mu J_\mu^\rho = \delta_\nu^\rho$. (An analogous calculation can be found in [15] for the case of a complex structure on a six dimensional manifold of Euclidean signature).

It is worthwhile mentioning that the vector $u^\mu = \bar{\lambda}\gamma^\mu \eta$ where λ is a commuting Majorana-Weyl spinor of the same chirality as η , is a null vector, i.e. $u^\mu u_\mu = 0$. Let there be N such vectors, $u_{(r)}^\mu$, $r = 1, \dots, N$. Then, the linear equation (31) can also be written as $u_{(r)}^\mu D_\mu \psi = 0$. Note also that a commuting Majorana-Weyl spinor η in 3+3 dimensions is pure, in the sense that $\bar{\eta}\eta = 0$.

Finally, using (24) and (25) we find that the self-duality condition (29) reduces to the following three equations

$$F_{13} = 0, \quad F_{16} = 0, \quad F_{36} = 0. \quad (35)$$

4. The KP Equation

The KP equation is a dynamical equation in 2+1 dimension and has the form

$$\partial_x \left(\dot{u} - \frac{1}{4}u_{xxx} - 3uu_x \right) = \frac{3}{4}u_{yy}, \quad (36)$$

where $u(x, y, t)$ is the dynamical variable, and subscripts x and y denote partial differentiations with respect to x and y , respectively. For a medium with an opposite dispersive behaviour, the equation takes the form

$$\partial_x \left(\dot{u} - \frac{1}{4}u_{xxx} - 3uu_x \right) = -\frac{3}{4}u_{yy}, \quad (37)$$

We shall first consider the embedding of the KP equation in the self-duality equations (26-28). To this end, we identify the coordinates x^1, x^3, x^6 with the coordinates t, x, y of a 2+1 dimensional spacetime and let

$$\begin{aligned} A_1 &= H, & A_3 &= Q, & A_6 &= \tilde{Q}, \\ A_2 &= -B, & A_4 &= P, & A_5 &= \tilde{B}. \end{aligned} \quad (38)$$

Next, we impose the reduction conditions

$$\partial_2 = 0, \quad \partial_4 - \partial_x = 0, \quad \partial_5 = 0, \quad (39)$$

and take B and \tilde{B} to be constant. With these conditions, the self-duality equations (26-28) reduce to

$$[\partial_t - H, \partial_x - Q] = 0, \quad (40)$$

$$[\partial_t - H, \partial_y - \tilde{Q}] = 0, \quad (41)$$

$$[\partial_x - Q, \partial_y - \tilde{Q}] = 0, \quad (42)$$

$$[P, B] = 0, \quad (43)$$

$$[P, \tilde{B}] = 0, \quad (44)$$

$$[B, \tilde{B}] = 0, \quad (45)$$

$$[H, B] = [\partial_x - Q, \partial_x - P] - [\tilde{Q}, \tilde{B}]. \quad (46)$$

Finally we need to make ansatz for the gauge fields Q , \tilde{Q} and B . As before, we take the gauge group to be $SL(2, \mathbb{R})$. Next, for Q and B we again take the ansatz (10), and in addition we take \tilde{B} to be proportional to B and \tilde{Q} proportional to Q . More specifically, we make the following ansatz

$$B = -\sigma_-, \quad Q = \lambda\sigma_3 - u\sigma_- + \sigma_+, \quad \tilde{B} = \alpha\sigma_-, \quad \tilde{Q} = \beta Q, \quad (47)$$

where α, β and λ are arbitrary constants. The consequences of these ansatz are as follows. From (42) we learn that

$$u_y = \beta u_x. \quad (48)$$

Consequently, (41) reduces to (40). Next, from (43-44) we see that

$$P_- = 0, \quad P_3 = 0. \quad (49)$$

As in the case of KdV, the constraint (46) determines H_- and H_3 :

$$\begin{aligned} H_- &= -(\alpha\beta + P_+), \\ H_3 &= -\frac{1}{2}(u + P_+)' - \lambda(\alpha\beta + P_+) \end{aligned} \quad (50).$$

Finally, from (40) we find that

$$(u + 2P_+)' = 0, \quad (51)$$

$$H_+ = u(\alpha\beta + P_+) - \lambda P_+' - \frac{1}{2}(u + P_+)'' , \quad (52)$$

$$\dot{u} = \frac{1}{2}(u + P_+)''' + (u - \alpha\beta - P_+)u' + 2\lambda^2 P_+' . \quad (53)$$

Note that (50) and (51-53) are related to (14-17) by shifting $P_+ \rightarrow P_+ + \alpha\beta$ in the latter set of equations. As before, choosing the integration constant in (51) to be zero, we obtain $P_+ = -\frac{1}{2}u$, and therefore all the components of the gauge potential are now determined, and (53) reduces to

$$\dot{u} - \frac{1}{4}u''' - \frac{3}{2}uu' = -(\lambda^2 + \alpha\beta)u' . \quad (54)$$

Differentiating this equation with respect to x and then using the constraint (48), we obtain

$$\partial_x \left(\dot{u} - \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x \right) = - \left(\frac{\lambda^2 + \alpha\beta}{\beta^2} \right) u_{yy} . \quad (55)$$

Rescaling $u \rightarrow 2u$ and setting $\frac{\lambda^2 + \alpha\beta}{\beta^2} = \pm \frac{3}{4}$, yields precisely the KP equations (36) and (37).

Given the ansatz (47) for the gauge potentials Q and \tilde{Q} , the results (50-52) for the Hamiltonian, it is clear that the vanishing curvature equation in 2+1 dimensions for an $SL(2, \mathbb{R})$ valued gauge potential yields the KP equation, as was shown in [16]. What we have found here is that the reduction scheme summarized in (39) and (47) is actually sufficient to determine the Hamiltonian and that the resulting dynamical equations are the KP equations. In other words, tuning the gauge potentials in spatial directions plus the self-duality condition determine the Hamiltonian and the dynamical field equations. If we were given only the vanishing curvature equations, we would have to make choices for the gauge potentials in spatial directions *and* the Hamiltonian to obtain the desired dynamical equations.

In fact, this is the situation we encounter in the case of the second-kind of self-duality condition (35), which happens to be the vanishing curvature condition; i.e. the ansatz (47) for the gauge potentials Q and \tilde{Q} , the results (50) and (51) for the Hamiltonian, which should themselves now be viewed as ansatz, of course, provide the embedding of the KP equation in the self-duality condition (35).

Going back to the ansatz (47), we saw that it gave rise to the constraint equation (48). As mentioned before, while this constraint is derived from the self-duality equation, it implies a 1 + 1 dimensional structure for the KP equation. We can, in fact, derive the KP equation

from self-duality conditions (40-46) without such a constraint. For example, let us consider the following alternative choice of the potentials

$$B = \alpha\sigma_-, \quad \tilde{B} = \beta\sigma_-, \quad Q = P = u\sigma_-, \quad \tilde{Q} = \frac{4\lambda}{3}\partial_y^{-1}\phi_x\sigma_-, \quad (56)$$

where u and ϕ are the dynamical variables and α, β, λ are arbitrary constants. Note that ∂_y^{-1} corresponds to an indefinite integration over the y -coordinate. Given this ansatz, we see that the self-duality equations (43-45) are trivially satisfied, and that (42) yields the result

$$\phi_{xx} = \frac{3}{4\lambda}u_{yy}, \quad (57)$$

and (46) is satisfied with

$$H_- = 0, \quad H_3 = 0. \quad (58)$$

Using (57), one finds that (41) is satisfied provided that (40) is satisfied. This is the dynamical equation

$$\dot{u} = \partial_x H_+. \quad (59)$$

It is clear that, in this case, we need to make an appropriate ansatz for H_+ in order to obtain the KP equation. This ansatz is

$$H_+ = \left(\frac{3}{2}u^2 + \frac{1}{4}u_{xx} + \lambda\phi\right)\sigma_-. \quad (60)$$

Indeed, substituting (60) into (59) and using (57) we obtain the KP equation (36). This derivation, in fact, is very similar in spirit to the original derivation of the KP equation [6]. We note, in this case, that the derivation would go through even if we assume the gauge group to be $U(1)$.

5. Conclusions

Contrary to the common belief that the KP equation would not result from the self-duality condition on the Yang-Mills potentials, we have explicitly derived these equations from self-duality conditions in 3+3 dimensions. The two types of self-duality conditions considered in this paper, we believe, are interesting to study further in their own right.

An interesting open question that remains open is the derivation of the KdV and/or KP *hierarchy* from the self-duality conditions. The presence of extra internal dimensions studied here, may indeed have relevance to this question. Another interesting open problem is to determine if choices less stringent than those in (47) on the gauge potentials would lead to new integrable systems in 2+1 or higher dimensions, and in any event to determine the class of integrable systems for which our self-duality conditions provide a unified framework.

ACKNOWLEDGEMENTS

We thank HoSeong La and Chris Pope for useful discussions. We are grateful to Steve Fulling for bringing to our attention the references on paracomplex numbers.

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